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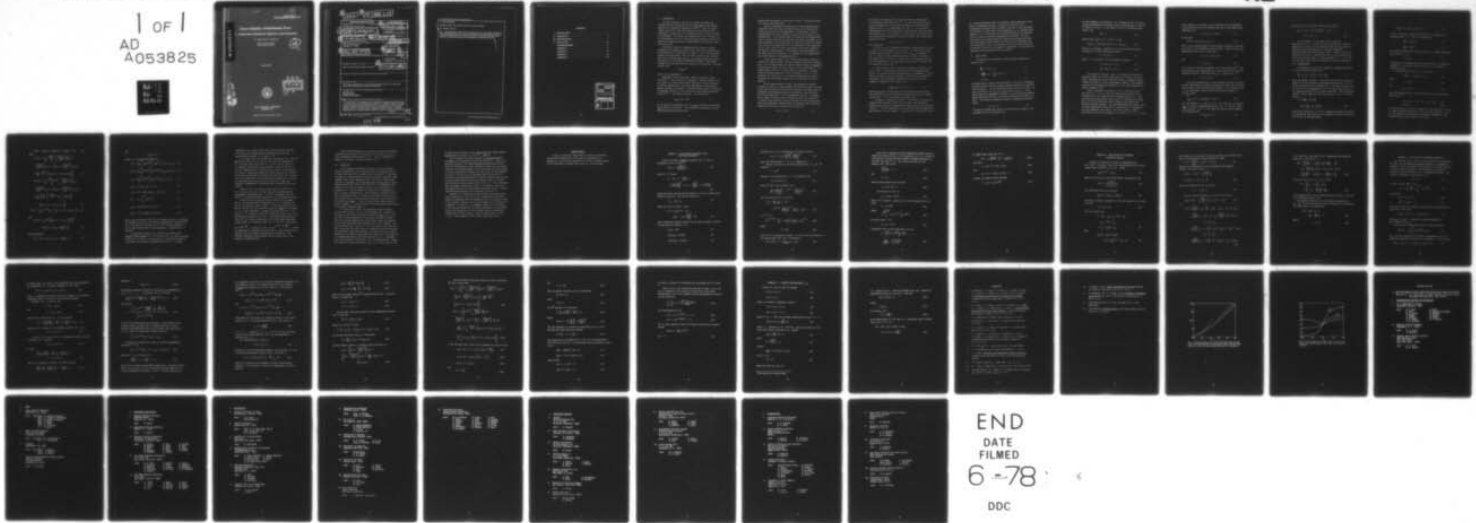
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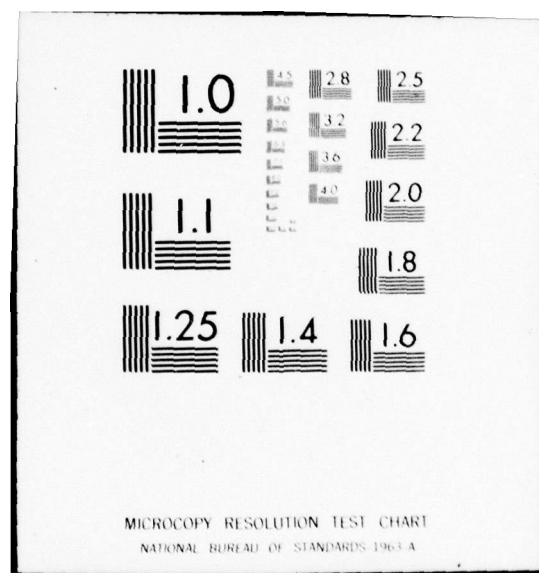
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Linear Stability of Self-Similar Flow:

1. Isothermal Cylindrical Implosion and Expansion

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*Plasma Dynamics Branch
Plasma Physics Division*

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20. Abstract (Continued)

liner. A complete solution is obtained for the perturbed motion. The only modes for which the perturbation amplitudes grow faster than the unperturbed liner radius are divergence- and curl-free. Numerical and analytical results are obtained for these and shown to reduce in the short wavelength limit to those found previously for incompressible time-independent basic states.

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I. INTRODUCTION

The Linus program at NRL has as its goal the design and demonstration of a fusion reactor utilizing compressional plasma heating by an imploding liquid metal liner.^[1,2] Use of a liquid (rather than solid) liner makes possible controlled repetitive implosions, in which most of the energy required to drive the liner is recoverable. This reduces the required value of Q , the ratio of the thermonuclear yield to the driving energy supplied, and also the effective dimensions and total energy needed for the device, since both scale as Q^2 .

One obvious problem in implementing the concept of plasma compression by cylindrical liquid liners stems from the possibility of Rayleigh-Taylor instabilities at radius $r = R_-(t)$, the interface between liner and plasma. When the liner slows and then rebounds, there is an effective gravity $g_{\text{eff}} \sim \ddot{R}_-$ pointing toward the center of the device, i.e., from the liner toward the plasma. Interchange instabilities localized near this surface can be expected to grow with typical rates

$$\gamma = (kg_{\text{eff}})^{\frac{1}{2}}, \quad (1)$$

where k is the wavenumber.

These modes can be stabilized, however, by means of liner rotation.^[2-5] Since viscosity is negligible (Reynolds numbers $\gg 10^5$ are typical for liquid alkali metals with scale sizes and implosion velocities characteristic of current experimental configurations), angular momentum is conserved. Thus each annular lamella of fluid has an azimuthal velocity $v \propto 1/R$, so that the centrifugal acceleration scales as R^{-3} . The effective gravity now has the form

$$g_{\text{eff}} = \ddot{R} - v^2/R. \quad (2)$$

If the second term dominates at $R = R_-$, $g_{\text{eff}} < 0$ there and the Rayleigh-Taylor modes are stabilized. This stabilization mechanism has been

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demonstrated experimentally using water liners^[5] and shown to agree with theoretical predictions.

Previous theoretical investigations of liner Rayleigh-Taylor instability have used fairly realistic models of the liner motion assuming the fluid to be incompressible. Nevertheless, non-zero compressibility can play a significant role in the evolution of the perturbations. If the growth time γ^{-1} derived for an incompressible fluid is shorter than the time required for sound to propagate a distance comparable to the wavelength, the growth rate should be reduced. The criterion for compressibility to be important is thus approximately $k \leq g/c^2$, where c is the sound speed in the liner (typically a few km/sec in liquid metals). For extremely short wavelengths, on the other hand, the formula [Eq. (1)] derived for an incompressible basic state should be a good approximation.

In the Linus-O device^[6] and in any larger device intended to operate in a reactor regime, the liner undergoes significant compression. This is particularly true near the inner surface at turnaround. Although such behavior has been studied in detail numerically,^[7] no simple analytic model of the basic motion is available. A linear stability analysis analogous to that of Barcilon, et al.,^[4] who described the evolution of the perturbation amplitudes in terms of integrals over the dynamic trajectories, is therefore out of the question.

In this paper we adopt a simple (and consequently somewhat unrealistic) model of the unperturbed liner motion in order to be able to get exact solutions to the stability problem. This model, that of isothermal self-similar contraction and expansion, was originally derived by Korobeinikov^[8] in an astrophysical context. In common with a number of related models developed by Sedov and coworkers,^[9] it yields a time-dependent nonlinear solution of the one-dimensional ideal fluid equations, starting with the assumption that in Eulerian variables the velocity u can be written in the form

$$u(r,t) = rF(t). \quad (3)$$

The density ρ and pressure p are then determined as functions of r^2 and quantities derivable from $F(t)$. In the case of the solution found by Korobeinikov, ρ and p are connected by an isothermal law (speed of sound $c = c_0$, independent of r and t). This is a good approximation for the liquid metals in Linus devices, where the maximum pressure $p_{\max} \sim 10^{10}$ dyne/cm² $\sim 0.1 \hat{\rho} c^2$ (here $\hat{\rho}$ is the density at zero pressure), and consequently the corrections to the lowest-order equation of state,

$$p = c_0^2 (\rho - \hat{\rho}), \quad (4)$$

are unimportant.

The basic state so determined describes a nonrotating liner. Thus we expect to find instability in all cases. (If sufficiently rapid rotation were introduced to change the sign of g_{eff} as defined in Eq. (2), stability would again be expected to occur.) The object of the present work is to see in a simple model how compressibility and the time dependence of the unperturbed motion modify the characteristic dispersion relation Eq. (1).

A word should be said here about the meaning of stability as we use the term. In analyzing perturbations about time-varying basic states, one in general finds non-exponential time dependence. We have chosen to regard a system as unstable if and only if the asymptotic development of the perturbed displacement ξ is faster than the asymptotic expansion of the system, i.e.,

$$\lim_{t \rightarrow \infty} | \xi(t) | / R(t) \rightarrow \infty, \quad (5)$$

where both ξ and R are associated with a particular fluid element. This definition does not preclude the possibility that in a "stable" system, $| \xi(t) | / R(t)$ increases for a finite time.

The paper is organized as follows. In Section II we obtain the general Sedov similarity solution for an ideal fluid obeying an adiabatic law, using the Lagrangian derivation previously described by Keller.^[10] We then specialize to the isothermal case and discuss its applicability to liner dynamics. In Section III we develop the theory

of a linear perturbation about an arbitrary time-varying basic state, again in Lagrangian variables. We apply this theory to the Korobeinikov isothermal model discussed in Section II and investigate the stability of flute modes (axial wavenumber $k_z = 0$, azimuthal mode number $m \neq 0$). We show that the resulting problem can be completely solved in terms of confluent hypergeometric functions. The only modes which are unstable during both implosion and expansion phases are associated with perturbations for which both the divergence and curl of $\underline{\xi}$ vanish. (Most of the details of the calculations described in Sections II and III are relegated to Appendices.) We conclude the paper with a brief discussion of our results.

II. BASIC STATE

We start with the continuity, force and state equations in Eulerian form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0; \quad (6)$$

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) + \nabla p = 0; \quad (7)$$

$$p = P(\rho). \quad (8)$$

We assume that the region within which Eqs. (6-8) hold is bounded by two free surfaces, where an external pressure supplied by a massless fluid balances p . Let the unperturbed motion of a fluid element, the position of which is \underline{r}_0 at $t=0$, be given by $\underline{R} = \underline{R}(\underline{r}_0, t)$. We can then transform to Lagrangian variables, i.e., write the displacement, density and radial velocity of the fluid element as functions of \underline{r}_0 and t . The similarity hypothesis becomes in Lagrangian variables simply

$$\underline{R} = \underline{r}_0 f(t). \quad (9)$$

We assume one-dimensional motion in the direction of symmetry, the coordinate of which is denoted by R .

We find (Appendix A) that when Eq. (4) is assumed for Eq. (8) and we introduce dimensionless variables, density being measured in units of ρ , velocity in units of c and length in units of the density scale length, then

$$\ddot{f}f = 1, \quad (10)$$

where we have taken $f(0) = 1$, $\dot{f}(0) = 0$, and

$$\rho(r_0, t) = (D/f) \exp(-r_0^2/2) = 1 + p(r_0, t), \quad (11)$$

where D is a constant. Equations (9-11) hold within the region occupied by the liner, defined according to

$$r_- \leq r_0 \leq r_+. \quad (12)$$

By Eq. (9) the motion of the two surfaces is given by

$$R_- = R(r_-, t) = r_- f; \quad (13)$$

$$R_+ = R(r_+, t) = r_+ f. \quad (14)$$

The similarity solution, Eqs. (9-11), should be viewed as a convenient model, not an exact description of the motion of a real liner. Nevertheless, we will show that there is a parameter regime in which the model is entirely appropriate, namely, in the immediate vicinity of the turning point reached by a thin liner compressing a plasma or magnetic flux.

Figure 1 displays the behavior of f as a function of time. Note that $f(-t) = f(t)$, so that the trajectories described by the similarity model are symmetric under reflection about $t = 0$. This contrast with the previously cited results^[7] determined by solving Eqs. (6-9) self-consistently with an adiabatic gas payload and vanishing external pressure. They indicated, among other things, that the liner expands more slowly than it contracts, due to tamping by the still-imploding outer layers. The latter effect is least noticeable for small peak pressures, i.e., when $p = \rho - 1 \ll 1$. For the

liner trajectory in our model to be consistent in the neighborhood of the turning point with an adiabatic gas law for the plasma being compressed, viz.,

$$p = \zeta (r_-/R_-)^{2\gamma} = \zeta f^{-2\gamma}, \quad (15)$$

we must have

$$\zeta f^{-2\gamma} \approx (D/f) \exp(-r_-^2/2) - 1. \quad (16)$$

Here ζ is the maximum plasma pressure reached, expressed in the same units ($\hat{\rho} c^2$) as the similarity solution itself. Using the early-time approximation (A22) for f we find that

$$D = (\zeta + 1) \exp(r_-^2/2) \quad (17)$$

and

$$\zeta = 1/(2\gamma - 1). \quad (18)$$

For a mixture of plasma and magnetic flux it has been shown^[11] that the pressure is well approximated by a law of the form (14) with an effective value of γ satisfying $5/3 < \gamma_{\text{eff}} < 2$, whence from (18)

$$1/4 < \zeta < 3/7. \quad (19)$$

The strict applicability of the similarity solution as a model for the liner motion is evidently limited by the fact that for sufficiently large f the pressure becomes negative. This occurs first at $r = r_+$, and defines a restriction on the similarity factor to $f \leq f_{\text{max}}$, where

$$f_{\text{max}} = (\zeta + 1) \exp[\frac{1}{2}(r_-^2 - r_+^2)]. \quad (20)$$

For thick liners, i.e., when $r_+^2 - r_-^2 > 2 \ln(1 + \zeta)$, Eq. (20) implies $f_{\text{max}} < 1$, which is inconsistent with Eq. (10). In the thin liner limit, when $r_+^2 - r_-^2 \ll 1$, the model can only be strictly applied for times such that

$$f \leq f_{\text{max}} \approx \zeta + 1, \quad (21)$$

and only then if the external pressure ζ_e satisfies

$$\zeta_e(t) = [(\zeta + 1)/f] \exp \left[\frac{1}{2}(r_-^2 - r_+^2) \right] - 1$$

$$\approx (\zeta + 1)/f - 1. \quad (22)$$

In the remainder of this paper we will relax the restrictions (19-22). Instead, we will concentrate on finding the complete time evolution of perturbations about the basic state, in the belief that their behavior will display the general features displayed by more realistic models. Support for this is provided by the result, established below, that the eigenfunctions of the most unstable modes are sharply peaked at $r = r_-$, i.e., far from the outer layers where the similarity solution is unphysical.

III. PERTURBATIONS

From Appendix B, we find the equation in Lagrangian variables for the perturbed displacement $\underline{\xi}$:

$$\ddot{\underline{\xi}} = \nabla_{\underline{R}} (\nabla_{\underline{R}} \cdot \underline{\xi} - \frac{1}{f^2} \underline{R} \cdot \underline{\xi}) + \frac{1}{f^2} \underline{\xi}, \quad (23)$$

where the gradient operators on the right hand side are defined with respect to the instantaneous unperturbed displacement \underline{R} . At this point we assume that $\underline{\xi}$ is independent of Z , the axial component of \underline{R} . As a result of this assumption, there is no fixed spatial scale in the problem associated with a parallel wavenumber. Thus the gradient operator ∇ which appears on the right hand side of Eq. (23) can be written

$$\nabla = \underline{e}_r \frac{\partial}{\partial R} + \underline{e}_\theta \frac{1}{R} \frac{\partial}{\partial \theta}$$

$$= \frac{1}{f} \left[\underline{e}_r \frac{\partial}{\partial r_0} + \underline{e}_\theta \frac{1}{r_0} \frac{\partial}{\partial \theta} \right] \quad (24)$$

We can then multiply Eq. (23) by f^2 , leaving an expression involving only operations with respect to the Lagrangian position (the position of $t=0$) and the azimuthal angle:

$$f^2 \ddot{\underline{\xi}} = \nabla (\nabla \cdot \underline{\xi} - \underline{r} \cdot \underline{\xi}) + \underline{\xi} \quad (25)$$

In Eq. (24) and the remainder of Section III, we suppress the subscript zero in \underline{r}_0 . The initial conditions

$$\underline{\xi}_0(\underline{r}) = \underline{\xi}(\underline{r}, 0), \quad (26)$$

$$\dot{\underline{\xi}}_0(\underline{r}) = \dot{\underline{\xi}}(\underline{r}, 0) \quad (27)$$

are arbitrary except that they must satisfy the boundary conditions which hold at $r = r_{\pm}$,

$$\nabla \cdot \underline{\xi} = 0, \quad (28)$$

a consequence of assuming that the external pressure is unperturbed (Appendix B). The problem defined by Eqs. (25-28) can now be completely solved.

We begin by rewriting Eq. (25) in the form

$$f^2 \ddot{\underline{\xi}} + \underline{r} \cdot \nabla \underline{\xi} = \nabla \sigma - \underline{r} \times \underline{\omega}, \quad (29)$$

where

$$\sigma = \nabla \cdot \underline{\xi} \quad (30)$$

and

$$\underline{\omega} = \nabla \times \underline{\xi}. \quad (31)$$

We can find equations for σ and $\underline{\omega}$ by operating on (29) with divergence and curl:

$$f^2 \ddot{\sigma} = \nabla^2 \sigma - \sigma - \underline{r} \cdot \nabla \sigma + \underline{r} \cdot \nabla \times \underline{\omega}; \quad (32)$$

$$f^2 \ddot{\underline{\omega}} = \underline{\omega}. \quad (33)$$

We first solve Eq. (33) for $\underline{\omega}$. Knowing $\underline{\omega}$, we can solve Eq. (32) for σ ; using the results found for $\underline{\omega}$ and σ , we finally solve Eq. (29) for $\underline{\xi}$. The details of the analysis are presented in Appendix C. The complete solution can be written in the form

$$\underline{\xi}(\underline{r}, t) = \underline{e}_r \alpha(\underline{r}, t) + \underline{e}_\theta \beta(\underline{r}, t) + \underline{e}_z \gamma(\underline{r}, t) + \nabla \phi, \quad (34)$$

where

$$\begin{aligned} \alpha(r, \theta, t) = \sum_{m,n} e^{im\theta} & \left[r^{\mu_n} \int_{r_1}^r \frac{dr W_n'(r)}{r^{\mu_n+1}} \left\{ \left[\sigma_n(0) - \frac{im \omega_n(0)}{\mu_n+1} \right] S_n(t) + \left[\dot{\sigma}_n(0) - \frac{im \dot{\omega}_n(0)}{\mu_n+1} \right] T_n(t) \right\} \right. \\ & \left. + \frac{im}{\mu_n+1} \frac{W_n(r)}{r} \left\{ \omega_n(0) f(t) + \dot{\omega}_n(0) g(t) \right\} \right]; \quad (35) \end{aligned}$$

$$\begin{aligned} \beta(r, \theta, t) = \sum_{m,n} e^{im\theta} & \left[im r^{\mu_n} \int_{r_2}^r \frac{dr W_n(r)}{r^{\mu_n+2}} \left\{ \left[\sigma_n(0) - \frac{im \omega_n(0)}{\mu_n+1} \right] S_n(t) + \left[\dot{\sigma}_n(0) - \frac{im \dot{\omega}_n(0)}{\mu_n+1} \right] T_n(t) \right\} \right. \\ & \left. + \frac{1}{\mu_n+1} \left\{ \frac{-m^2}{r} \int_{r_3}^r \frac{dr W_n(r)}{r} + \frac{1}{r} \int_{r_3}^r dr W_n(r) \right\} \right. \\ & \left. \left\{ \omega_n(0) f(t) + \dot{\omega}_n(0) g(t) \right\} \right]; \quad (36) \end{aligned}$$

$$\gamma(r, \theta, t) = \sum_{m,n} e^{im\theta} W_n(r) \left\{ \gamma_n(0) f(t) + \dot{\gamma}_n(0) g(t) \right\}; \quad (37)$$

and

$$\begin{aligned} \phi(r, \theta, t) = \sum_m e^{im\theta} & \left\{ [A_{-p_-}(t) + B_{-q_-}(t)] \left(\frac{r_-}{r} \right)^m \right. \\ & \left. + [A_{+p_+}(t) + B_{+q_+}(t)] \left(\frac{r}{r_+} \right)^m \right\}. \quad (38) \end{aligned}$$

Here W_n satisfies

$$W_n'' + \left(\frac{1}{r} - r \right) W_n' + \left(\mu_n - 1 - \frac{m^2}{r^2} \right) W_n = 0 \quad (39)$$

and

$$W_n(r_{\pm}) = 0,$$

where μ_n is an associated eigenvalue;

$$\sigma_n(t) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-im\theta} \int_{r_-}^{r_+} dr r e^{-r^2/2} W_n(r) \sigma(r, \theta, t); \quad (40)$$

$$\omega_n(t) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-im\theta} \int_{r_-}^{r_+} dr r e^{-r^2/2} W_n(r) \underline{e}_z \cdot \underline{\omega}(r, \theta, t); \quad (41)$$

$$\gamma_n(t) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-im\theta} \int_{r_-}^{r_+} dr r e^{-r^2/2} W_n(r) \gamma(r, \theta, t); \quad (42)$$

$$S_n(t) = \Phi \left[-\frac{1}{2} \mu_n, \frac{1}{2}; \ln f(t) \right]; \quad (43)$$

$$T_n(t) = \dot{f}(t) \Phi \left[\frac{1}{2} - \frac{1}{2} \mu_n, \frac{3}{2}; \ln f(t) \right]; \quad (44)$$

$$g(t) = f(t) \int_0^{\dot{f}(t)} ds e^{-s^2/2}; \quad (45)$$

$$p_{\pm}(t) = \Phi \left[\frac{1}{2} (1 \mp m), \frac{1}{2}; \ln f(t) \right]; \quad (46)$$

$$q_{\pm}(t) = \dot{f}(t) \Phi \left[1 \mp \frac{1}{2} m, \frac{3}{2}; \ln f(t) \right], \quad (47)$$

and A_{\pm} , B_{\pm} are constants. Here Φ is the standard solution of the first kind (Kummer function) of the confluent hypergeometric equation. ^[13, 14]

All of the quantities defined in Eqs. (40-47) except g depend on m , but the mode number label has been suppressed to streamline the notation.

To investigate stability, we consider the implosion and expansion phases separately. First we note that f , S_n and p_{\pm} are even functions of t , while g , T_n and q_{\pm} are odd. Solutions of pure even (odd) parity are symmetric (antisymmetric) about $t=0$. A linear

combination, e.g., Q [Eq. (C51)] of the two can be chosen such that the solution is a monotonically increasing function of time asymptotically for both $t \rightarrow -\infty$ and $t \rightarrow +\infty$.

We consider first the outward motion starting at $t=0$. From the asymptotic formula (C54) we see that the only time dependence in Eq. (34) which grows faster than $f(t)$ is that in $p_-(t)$ and $q_-(t)$. [In Appendix D we show that this statement is true for all $t > 0$, not just asymptotically.] While ξ_- could be small or even vanish at $t=0$, owing to cancellation among the various terms in the solution, so that the perturbations could grow over some finite time interval, the relative amplitude of the perturbations vanishes as $t \rightarrow \infty$ except for modes which have finite A_- or B_- in Eq. (38). These are evidently incompressible irrotational modes localized at the inner surface.

From Eq. (C54), p_-/f and q_-/f both grow as $(\ln f)^{m/2}$ as $t \rightarrow \infty$. For t close to zero, Eq. (C25) says that the perturbations grow exponentially with growth rate $\gamma = (m-1)^{1/2}$, in agreement with the result^[4] for Rayleigh-Taylor instability in an incompressible liner.

Now we turn to the question of the stability of the inward motion. We ask under what circumstances all of the components of ξ_-/f can have smaller values at some $t < 0$ than they attain at turn-around. We are thus led to seek solutions which damp with increasing $-t$, i.e., with increasing f . Evidently they must have $\dot{\omega}(0) = -\omega(0)$ and $\dot{\gamma}(0) = -\gamma(0)$ and must contain the functions p_- and q_- only in the combination Q , defined in Eq. (C51). The fastest growing modes are those with all constant coefficients set to zero except those multiplying S_n and T_n . According to Eqs. (C26-27) and (C54), these grow as $(\ln f)^{\frac{1}{2} + \frac{\mu n}{2}}$, or using Eq. (C24), as $(\ln f)^{m + \frac{1}{2} + \frac{1}{2} \left(\frac{n\pi}{\Delta r}\right)^2}$. Clearly the amplification increases with both n and m . These modes are very different from the familiar Rayleigh-Taylor instability. They represent overstable internal waves, and are entirely due to the presence of compressibility.

If we look for modes which are unstable both before and after turnaround, we see that the only ones are those with $\gamma = \underline{\omega} = \sigma = 0$ and time dependence proportional to $Q(t)$. Figure 2 compares the behavior of Q^m for $m = 1$ and 10 with that of f .

IV. CONCLUSION

Using Lagrangian coordinates throughout, we have obtained general formulations of the one-dimensional self-similar motion of an ideal isentropic fluid (already in the literature^[10]) and of the development of infinitesimal perturbations around an arbitrary time-dependent basic state (apparently new). The two are combined and applied to a treatment of the Rayleigh-Taylor instability in a self-similar fluid motion. When, as is true here for z -independent perturbations, the gradient operator acting on the perturbations is homogeneous in r , we can separate the space and time dependence of $\xi(\underline{r}, t)$, a great simplification. Thus the analysis presented here of flute perturbations about a cylindrical self-similar motion is actually simpler than the corresponding slab problem, which involves the perturbation wavelength. In fact, the spherical problem, which is treated elsewhere,^[15] can be solved for all modes without restriction on the angular dependence $Y_l^m(\theta, \varphi)$.

For liquid metal liners, the appropriate relation between pressure and density is Eq. (4), the isothermal law. Carrying out the analysis, we obtain the complete solution for $\xi(\underline{r}, t)$, Eq. (34). One class of modes grows prior to turnaround and damps afterward. These are internal compressional (sound) waves, which are pumped up by the liner contraction and relax again as the liner expands. The only modes which grow (relative to the liner radius) both before and after turnaround are those which are incompressible and irrotational. For sufficiently large mode numbers m , or times sufficiently close to turnaround, the growth is exponential and identical to that found for an incompressible medium, essentially given by Eq. (1). Otherwise their time dependence, given by Eq. (C51), is nonexponential. Higher mode numbers m (shorter wavelengths) give rise to faster growth, with

the amplification relative to the unperturbed liner radius scaling asymptotically for large t as $(\ln f)^m \sim (t\sqrt{\ln t})^m$.

This is similar to results obtained in magnetohydrodynamic treatments of the stability of static equilibria.^[12] There it is generally found that the fastest-growing perturbations are those which are incompressible, since compression reduces the free energy available for producing magnetic field deformation and fluid motion. It is also interesting to compare our result with the well-known^[16] solution to the problem of the Rayleigh-Taylor instability of a stationary isothermal exponentially stratified medium supported from below by another medium of much lower density, where it is found that the growth rates are given exactly by Eq. (1). This conclusion, which may appear surprising in light of our arguments (Section I) that compressibility can reduce growth rates when $k \leq g/c^2$, finds an explanation when reached by the techniques of Section II.

Equation (28) states that at free surfaces the medium deforms without compression. Since in a static exponentially stratified medium the eigenfunctions vary exponentially in the direction normal to the boundary, condition (28) holds everywhere if it holds on the boundaries. In effect, it is propagated inward from the surfaces. In the problem treated here, the radial dependence of the perturbations is not exponential, but the unstable modes are essentially localized at the inner surface. Hence we should expect to find $\nabla \cdot \underline{\xi} = 0$, even though the model allows compressible motion.

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APPENDIX A. SELF-SIMILAR SOLUTIONS OF THE IDEAL FLUID EQUATIONS

When we introduce Lagrangian variables, Eq. (6) can be integrated at once to yield^[12]

$$\rho(\underline{r}_0, t) = \frac{\rho(\underline{r}_0, 0)}{|\nabla_{\underline{R}} \underline{r}_0(t)|}, \quad (A1)$$

while Eq. (7) becomes

$$\begin{aligned} \ddot{\underline{R}} &= -\frac{1}{\rho} \nabla p = -\frac{1}{\rho} \frac{dp}{d\rho} \nabla \rho \\ &= -\frac{1}{\rho} \frac{dp}{d\rho} \frac{(\nabla_{\underline{R}} \underline{r}_0)^A}{|\nabla_{\underline{R}} \underline{r}_0|} \cdot \nabla_{\underline{O}} \rho = -\frac{(\nabla_{\underline{R}} \underline{r}_0)^A}{|\nabla_{\underline{R}} \underline{r}_0|} \cdot \nabla_{\underline{O}} \int \frac{dp}{\rho}, \end{aligned} \quad (A2)$$

where dots denote derivatives with respect to t and the superscript A denotes the adjoint. Here we have made use of

$$\nabla \underline{r}_0 \cdot \nabla_{\underline{O}} \underline{R} = \underline{\underline{I}}, \quad (A3)$$

where $\underline{\underline{I}}$ is the unit dyadic. Hence

$$\begin{aligned} \nabla \rho &= [\nabla \underline{r}_0(\underline{R}, t)] \cdot \nabla_{\underline{O}} \rho = \\ &= (\nabla_{\underline{R}} \underline{r}_0)^{-1} \cdot \nabla_{\underline{O}} \rho = \frac{(\nabla_{\underline{R}} \underline{r}_0)^A}{|\nabla_{\underline{R}} \underline{r}_0|} \cdot \nabla_{\underline{O}} \rho. \end{aligned} \quad (A4)$$

These expressions simplify greatly for the cases of planar, cylindrical and spherical symmetry, viz.,

$$\rho_0 dr_0 = \rho dR; \quad (A5)$$

$$2\pi \rho_0 r_0 dr_0 = 2\pi \rho R dR; \quad (A6)$$

$$4\pi \rho_0 r_0^2 dr_0 = 4\pi \rho r^2 dR. \quad (A7)$$

Equations (A5-7) can be summarized in the single relation

$$\rho(r, t) = \rho(r_0, 0) \left(\frac{r_0}{R} \right)^{\nu-1} \left(\frac{\partial R}{\partial r_0} \right)^{-1}, \quad (A8)$$

where the dimensionality $\nu = 1, 2, 3$, respectively.

When we introduce the similarity assumption Eq. (9), Eq. (A8) reduces to

$$\rho = \rho_0 f^{-\nu} \quad (A9)$$

Moreover, if we specialize Eq. (8) to an adiabatic law

$$p = \hat{p} [(\rho/\hat{\rho})^\gamma - 1], \quad (A10)$$

where \hat{p} , $\hat{\rho}$ and γ are constants, then

$$\frac{dp}{d\rho} = \frac{\hat{p}}{\hat{\rho}} \left(\frac{\rho}{\hat{\rho}} \right)^{\gamma-1} = f^{-\nu(\gamma-1)} \frac{\hat{p}}{\hat{\rho}} \left(\frac{\rho_0}{\hat{\rho}} \right)^{\gamma-1}, \quad (A11)$$

and one can write (A2) in the form

$$\begin{aligned} r_0 \ddot{f} = & - \frac{dp}{d\rho} \frac{\partial}{\partial R} \ln(\rho/\hat{\rho}) = \\ & - f^{-\nu(\gamma-1)} \frac{\hat{p}}{\hat{\rho}} \left(\frac{\rho_0}{\hat{\rho}} \right)^{\gamma-1} \frac{1}{f} \frac{\partial}{\partial r_0} \ln(\rho_0/\hat{\rho}), \end{aligned} \quad (A12)$$

or if $\gamma \neq 1$,

$$\ddot{f} f^{\nu(\gamma-1)+1} = \frac{c^2}{\ell^2} = - \frac{\gamma}{\gamma-1} \frac{\hat{p}}{\hat{\rho}} \frac{\partial}{\partial (r_0^{2/2})} (\rho_0/\hat{\rho})^{\gamma-1}, \quad (A13)$$

where

$$c^2 = \hat{p}/\hat{\rho} \quad (A14)$$

and $\ell^2 > 0$ is a separation constant. If $\gamma=1$ (the case of interest in the present work), Eq. (A13) is replaced by

$$\ddot{f} f = c^2/\ell^2 = - c^2 \frac{\partial}{\partial (r_0^{2/2})} \ln(\rho_0/\hat{\rho}). \quad (A15)$$

Note that by assuming a positive separation constant, we have effectively chosen case III (in the notation of ref. 9), in which all fluid particle trajectories pass through a minimum displacement at the same time (when $\dot{f} = 0$). This instant we denote by $t=0$. Equation (A15) thus yields

$$\frac{d \ln \rho_0}{d(r_0^2/2)} = -1. \quad (\text{A16})$$

and

$$\ddot{f} = 1/f, \quad (\text{A17})$$

where we have adopted units such that

$$c = l = \hat{\rho} = \hat{p} = 1. \quad (\text{A18})$$

The solution of (A16) is

$$\rho_0 = D \exp(-r_0^2/2), \quad (\text{A19})$$

where D is a constant. Equation (A17) can be integrated once to yield

$$\dot{f} = (2 \ln f)^{\frac{1}{2}}, \quad (\text{A20})$$

whence

$$t = \int_0^{\sqrt{2 \ln f}} ds e^{s^2/2} = \int_0^{\dot{f}} ds e^{s^2/2}. \quad (\text{A21})$$

For early times $t \ll 1$,

$$f(t) \approx 1 + \frac{1}{2} t^2. \quad (\text{A22})$$

Integrating (A21) by parts repeatedly, we find

$$\begin{aligned} t &= \frac{e^{\dot{f}^2/2} - 1}{\dot{f}} + \frac{e^{\dot{f}^2/2} - 1}{\dot{f}^3} - \frac{1}{2} \frac{\dot{f}^2}{\dot{f}^3} + \dots \\ &= \frac{f-1}{\sqrt{2 \ln f}} + \frac{f-1 + \ln f}{(2 \ln f)^{\frac{3}{2}}} + \dots \end{aligned} \quad (\text{A23})$$

To lowest order, then, for $t \gg 1$

$$f(t) \sim t \sqrt{2 \ln t} \left[1 + O\left(\frac{1}{\ln t}\right) \right]. \quad (\text{A24})$$

From (A9),

$$\rho = \rho_0 / f = f^{-1} \exp(-r_0^2/2) \quad (\text{A25})$$

and

$$p = \rho - 1 = f^{-1} \exp(-r_0^2/2) - 1. \quad (\text{A26})$$

Finally, the radial velocity becomes

$$u = r_0 \dot{f} = r_0 (2 \ln f)^{\frac{1}{2}}. \quad (\text{A27})$$

APPENDIX B. DERIVATION OF THE PERTURBED EQUATION OF MOTION

Consider the equations of motion in Lagrangian form, Eqs. (A1-2). Let $\underline{R}(\underline{r}_0, t)$ denote the unperturbed motion of a fluid element whose location is \underline{r}_0 at $t=0$. Hence

$$\rho_0(\underline{R}, t) \ddot{\underline{R}} = - \nabla_{\underline{R}} p_0, \quad (B1)$$

where ρ_0 and p_0 are the unperturbed density and pressure, and

$$\rho(\underline{R}, t) = \frac{\rho_0(\underline{r}_0, 0)}{|\nabla_{\underline{r}_0} \underline{R}(\underline{r}_0, t)|}. \quad (B2)$$

Let a perturbed motion be defined by

$$\underline{r}(\underline{r}_0, t) = \underline{R}(\underline{r}_0, t) + \underline{\xi}(\underline{r}_0, t). \quad (B3)$$

Note that on taking the gradient of (B3) with respect to \underline{r} at fixed t one has

$$\underline{\underline{I}} = \nabla \underline{R} + \nabla \underline{\xi}. \quad (B4)$$

But by the chain rule

$$\begin{aligned} \nabla &= (\nabla \underline{R}) \cdot \nabla_{\underline{R}} = (\underline{\underline{I}} - \nabla \underline{\xi}) \cdot \nabla_{\underline{R}} \\ &= \nabla_{\underline{R}} - (\nabla \underline{\xi}) \cdot \nabla_{\underline{R}} \\ &\approx \nabla_{\underline{R}} - (\nabla_{\underline{R}} \underline{\xi}) \cdot \nabla_{\underline{R}}. \end{aligned} \quad (B5)$$

Thus

$$\begin{aligned} |\nabla_{\underline{R}} \underline{r}| &= |\nabla_{\underline{R}} \underline{R} + \nabla_{\underline{R}} \underline{\xi}| \\ &= |\underline{\underline{I}} + \nabla_{\underline{R}} \underline{\xi}| \approx 1 + \nabla_{\underline{R}} \cdot \underline{\xi}. \end{aligned} \quad (B6)$$

The integral of the perturbed continuity equation (the condition that mass is conserved under the perturbation B3) reads

$$\rho(\underline{r}, t) = \frac{\rho_o(\underline{R}, t)}{|\nabla_{\underline{R}} \underline{r}|} = \rho_o(\underline{R}, t) + \rho_1(\underline{R}, t), \quad (B7)$$

where we use \underline{R} to identify a given Lagrangian fluid element.

Together (B6) and (B7) yield

$$\begin{aligned} \rho_o(\underline{R}, t) + \rho_1(\underline{R}, t) &= \frac{\rho_o(\underline{R}, t)}{1 + \nabla_{\underline{R}} \cdot \underline{\xi}} \\ &\approx \rho_o(\underline{R}, t) - \rho_o(\underline{R}, t) \nabla_{\underline{R}} \cdot \underline{\xi} \end{aligned} \quad (B8)$$

Hence the perturbed density is given by

$$\rho_1 = - \rho_o \nabla_{\underline{R}} \cdot \underline{\xi} \quad (B9)$$

Now the linearized equation of motion can be written

$$\rho_o \ddot{\underline{\xi}} + \rho_1 \ddot{\underline{R}} = - \nabla_{\underline{R}} \left[\frac{dp_o}{d\rho_o} \rho_1 \right] + (\nabla_{\underline{R}} \underline{\xi}) \cdot \nabla_{\underline{R}} p_o \quad (B10)$$

or

$$\begin{aligned} \rho_o \ddot{\underline{\xi}} &= \nabla_{\underline{R}} \left[\frac{dp_o}{d\rho_o} \rho_o \nabla_{\underline{R}} \cdot \underline{\xi} \right] + (\nabla_{\underline{R}} \underline{\xi}) \cdot \nabla_{\underline{R}} p_o - (\nabla_{\underline{R}} \cdot \underline{\xi}) \nabla_{\underline{R}} p_o \\ &= \nabla_{\underline{R}} \left[\frac{dp_o}{d\rho_o} \nabla_{\underline{R}} \cdot (\rho_o \underline{\xi}) - \underline{\xi} \cdot \nabla_{\underline{R}} \rho_o \frac{dp_o}{d\rho_o} \right] + \nabla_{\underline{R}} \underline{\xi} \cdot \nabla_{\underline{R}} p_o \\ &\quad - (\nabla_{\underline{R}} \cdot \underline{\xi}) \nabla_{\underline{R}} p_o \\ &= \nabla_{\underline{R}} \left[\frac{dp_o}{d\rho_o} \nabla_{\underline{R}} \cdot (\rho_o \underline{\xi}) \right] - \nabla_{\underline{R}} [\underline{\xi} \cdot \nabla_{\underline{R}} p_o] + (\nabla_{\underline{R}} \underline{\xi}) \cdot \nabla_{\underline{R}} p_o \\ &\quad - (\nabla_{\underline{R}} \cdot \underline{\xi}) \nabla_{\underline{R}} p_o \\ &= \nabla_{\underline{R}} \left[\frac{dp_o}{d\rho_o} \nabla_{\underline{R}} \cdot (\rho_o \underline{\xi}) \right] - \underline{\xi} \cdot \nabla_{\underline{R}} \nabla_{\underline{R}} p_o - \nabla_{\underline{R}} \cdot \underline{\xi} \nabla_{\underline{R}} p_o \end{aligned} \quad (B11)$$

If we divide by ρ_0 and rewrite (B11), suppressing the subscripts \underline{R} and zero, there results

$$\begin{aligned}\ddot{\underline{\xi}} &= \nabla \left[\frac{1}{\rho} \frac{dp}{d\rho} \nabla \cdot (\rho \underline{\xi}) \right] + \frac{1}{\rho^2} \nabla \rho \frac{dp}{d\rho} \nabla \cdot (\rho \underline{\xi}) \\ &\quad - \underline{\xi} \cdot \nabla \left[\frac{1}{\rho} \nabla p \right] - \frac{1}{\rho^2} \underline{\xi} \cdot \nabla \rho \nabla p - \frac{1}{\rho} (\nabla \cdot \underline{\xi}) \nabla p \\ &= \nabla \left[\frac{1}{\rho} \frac{dp}{d\rho} \nabla \cdot (\rho \underline{\xi}) \right] - \underline{\xi} \cdot \nabla \left[\frac{dp}{d\rho} \nabla \ln \rho \right] \quad (B12)\end{aligned}$$

Using (A25) and (A26) for ρ and p , we find

$$\begin{aligned}\ddot{\underline{\xi}} &= \nabla \left[\frac{1}{\rho} \nabla \cdot (\rho \underline{\xi}) \right] + \frac{1}{f^2} \underline{\xi} \\ &= \nabla \left[\nabla \cdot \underline{\xi} - \frac{\underline{R} \cdot \underline{\xi}}{f^2} \right] + \frac{\underline{\xi}}{f^2}, \quad (B13)\end{aligned}$$

the Lagrangian form of the perturbed equation of motion in the case of an isothermal pressure law.

The boundary condition at a free surface associated with Eq. (B12) is found from requiring that the pressure on the boundary not change under the perturbations:

$$\begin{aligned}p(\underline{R}) &= p_0(\underline{R}) + p_1(\underline{R}) \\ &= p_0(\underline{R}) - \frac{dp}{d\rho} \rho_0 \nabla \cdot \underline{\xi} \quad (B14)\end{aligned}$$

whence

$$\nabla \cdot \underline{\xi} = 0 \quad (B15)$$

APPENDIX C. SOLUTION OF THE PERTURBATION EQUATION

The solution of Eqs. (25-28) is facilitated by two circumstances: the space and time dependence separate completely (in Lagrangian variables), and the solutions can be carried out in stages so that ξ is found "a piece at a time." The first piece is the position vorticity \underline{w} , which satisfies Eq. (33), a homogeneous equation. One solution is immediate, since from Eq. (10),

$$f^2 \ddot{f} = f. \quad (C1)$$

The other is then found to be

$$g(t) = f \int_0^t \frac{dt}{f^2} = f \int_0^{\dot{f}(t)} ds e^{-s^2/2}. \quad (C2)$$

We thus have

$$f(0) = 1, \dot{f}(0) = 0; \quad (C3)$$

$$g(0) = 0, \dot{g}(0) = 1, \quad (C4)$$

and so we can write the complete solution of (33) including initial conditions, in the form

$$\underline{w}(\underline{r}, t) = \underline{w}_0(\underline{r}) f(t) + \dot{\underline{w}}_0(\underline{r}) g(t), \quad (C5)$$

where $\underline{w}_0(\underline{r}) = \underline{w}(\underline{r}, 0)$. Evidently the spatial dependence of \underline{w} is "frozen in" at $t=0$.

We now Fourier transform ξ , \underline{w} and σ in angle according to

$$\xi^m(\underline{r}, t) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-im\theta} \xi(\underline{r}, t), \quad (C6)$$

etc. [In what follows we will display the dependence on m in the transformed quantities only where it is important.] If we now write

$$\xi(\underline{r}, t) = \underline{e}_r \alpha(\underline{r}, t) + \underline{e}_\theta \beta(\underline{r}, t) + \underline{e}_z \gamma(\underline{r}, t), \quad (C7)$$

we observe that, as a result of our assumption that the perturbations are independent of z , the radial component of (C5) yields

$$\gamma(r, t) = \gamma_0(r) f(t) + \dot{\gamma}_0(r) g(t). \quad (C8)$$

Since γ is completely determined and does not appear in the equations governing α and β , we can omit it from further discussion.

Writing $\omega = \underline{e}_z \cdot \underline{\omega}$, we note that

$$\underline{r} \times \underline{\omega} = -\underline{e}_\theta r\omega \quad (C9)$$

and

$$\underline{r} \cdot \nabla \times \underline{\omega} = im\omega \quad (C10)$$

After Fourier transforming, Eq. (32) thus becomes

$$f^2 \ddot{\sigma} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \sigma}{\partial r} \right) - r \frac{\partial \sigma}{\partial r} - \left(1 + \frac{m^2}{r^2} \right) \sigma + im\omega. \quad (C11)$$

Equation (C11) is subject to the boundary condition (B15), i.e.,

$$\sigma(r_-) = \sigma(r_+) = 0 \quad (C12)$$

To solve (C11), consider first the homogeneous equation obtained by setting $\omega = 0$. Writing

$$\sigma(r, t) = W(r) S(t), \quad (C13)$$

we find

$$\begin{aligned} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dW}{dr} \right) - r \frac{dW}{dr} - \left(1 + \frac{m^2}{r^2} \right) W \right] W^{-1} = \\ = -\mu = f^2 \ddot{S} S^{-1}, \end{aligned} \quad (C14)$$

where μ is a separation constant. The radial equation

$$\frac{d^2 W}{dr^2} + \left(\frac{1}{r} - r \right) \frac{dW}{dr} + \left(\mu - 1 - \frac{m^2}{r^2} \right) W = 0, \quad (C15)$$

subject to

$$W(r_{\pm}) = 0, \quad (C16)$$

determines an eigenvalue problem with solutions W_n corresponding to μ_n , $n = 1, 2, \dots$. Writing Eq. (C15) in the form

$$\frac{1}{r} \frac{d}{dr} \left[r e^{-r^2/2} \frac{dW}{dr} \right] + \left[\mu - 1 - \frac{m^2}{r^2} \right] e^{-r^2/2} W = 0, \quad (C17)$$

we see that

$$\mu = \frac{\int_{r_-}^{r_+} dr r e^{-r^2/2} \left[\left(\frac{dW}{dr} \right)^2 + \left(1 + \frac{m^2}{r^2} \right) W^2 \right]}{\int_{r_-}^{r_+} dr r e^{-r^2/2} W^2} > 0, \quad (C18)$$

so that the eigenfunctions are oscillatory. From classical Sturm-Liouville theory, we know that the eigenfunctions $W_n(r)$ form a complete orthonormalizable basis in the space of real functions on (r_-, r_+) , with the orthogonality condition given by

$$\int_{r_-}^{r_+} dr r e^{-r^2/2} W_n(r) W_{n'}(r) = \delta_{nn'}. \quad (C19)$$

The W 's can be expressed in terms of confluent hypergeometric functions,^[13] viz.,

$$W_n(r) = r^{\pm m} F \left[\frac{1}{2} (1 \pm m - \mu_n), 1 \pm m; \frac{1}{2} r^2 \right], \quad (C20)$$

where $F(a, c; x)$ is a solution of

$$x \frac{d^2 F}{dx^2} + (c - x) \frac{dF}{dx} - a F = 0. \quad (C21)$$

The μ 's must in general be determined numerically. The most efficient method is to utilize (C18) as a variational principle and solve for μ by the Rayleigh-Ritz procedure. Given μ , a linear combination of

two independent solutions of (C21) can be found which satisfies (C12). The asymptotic forms of the standard solutions of the confluent hypergeometric equation (C21) in the limit $a \rightarrow -\infty$ for fixed x are^[13]

$$\begin{aligned} \Phi(a, c; x) \sim \pi^{-\frac{1}{2}} \Gamma(c) \left(\frac{1}{2} cx - ax\right)^{\frac{1}{4} - \frac{c}{2}} \exp\left(\frac{1}{2}x\right) \\ \cdot \cos \left\{ \left[(2c - 4a)x\right]^{\frac{1}{2}} - \frac{1}{2} \pi c + \frac{1}{4} \pi \right\} \end{aligned} \quad (C22)$$

and

$$\begin{aligned} \Psi(a, c; x) \sim \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}c - a + \frac{1}{4}\right) x^{\frac{1}{4} - \frac{c}{2}} \exp\left(\frac{1}{2}x\right) \\ \cdot \cos \left\{ \left[(2c - 4a)x\right]^{\frac{1}{2}} + \left(a - \frac{1}{2}c + \frac{1}{4}\right) \pi \right\}. \end{aligned} \quad (C23)$$

Using these we can calculate the asymptotic behavior of the eigenvalues μ_n , i.e., the WKB approximation for μ_n . The result is

$$\mu_n = 2m + \left(\frac{n\pi}{\Delta r}\right)^2, \quad (C24)$$

$n = 1, 2, \dots$, where $\Delta r = r_+ - r_-$. All but the first few eigenvalues are well approximated by (C24), and the approximation improves as $\Delta r \rightarrow 0$.

The time dependence is given by

$$f^2 \ddot{S} + \mu_n S = 0. \quad (C25)$$

Using $\ln f$ as the new independent variable, we can convert (C25) into a confluent hypergeometric equation also. The result is

$$S(t) = F\left(-\frac{1}{2}\mu_n, \frac{1}{2}; \ln f\right) - f^{\frac{1}{2}} D_{-\mu_n}(\dot{f}), \quad (C26)$$

where D is a solution of the parabolic cylinder (Weber-Hermite) equation. It is convenient to choose two independent solutions of (C26) as

$$S_n(t) = \Phi\left(-\frac{\mu_n}{2}, \frac{1}{2}; \ln f\right), \quad (C27)$$

$$T_n(t) = \dot{f}(t) \Phi\left(\frac{1}{2} - \frac{\mu_n}{2}, \frac{3}{2}; \ln f\right), \quad (C28)$$

where Φ is the Kummer function^[13] associated with (C21). From this choice it follows that

$$S(0) = 1, \dot{S}(0) = 0; \quad (C29)$$

$$T(0) = 0, \dot{T}(0) = 1. \quad (C30)$$

We next seek a particular solution of the inhomogeneous equation (C11) in the form

$$\tilde{\sigma}(r, t) = \sum_n \tilde{\sigma}_n(t) W_n(r). \quad (C31)$$

Using (C5) and C15), we find

$$f^2 \ddot{\tilde{\sigma}}_n + \mu_n \tilde{\sigma}_n = im [\omega_n(0) f(t) + \dot{\omega}_n(0) g(t)]. \quad (C32)$$

The desired particular solution is thus simply

$$\tilde{\sigma}_n = \frac{im}{\mu_n + 1} [\omega_n(0) f + \dot{\omega}_n(0) g], \quad (C33)$$

and the complete solution, including initial conditions, is

$$\begin{aligned} \sigma_n(t) = & \left[\sigma_n(0) - \frac{im \omega_n(0)}{\mu_n + 1} \right] S_n(t) + \\ & \left[\dot{\sigma}_n(0) - \frac{im \dot{\omega}_n(0)}{\mu_n + 1} \right] T_n(t) + \frac{im}{\mu_n + 1} [\omega_n(0) f(t) \\ & + \dot{\omega}_n(0) g(t)]. \end{aligned} \quad (C34)$$

Now we can find a particular solution of (29) in the form $\tilde{\mathbf{E}} = \mathbf{e}_r \alpha + \mathbf{e}_\theta \beta$, where

$$\begin{aligned} \alpha(r, t) = \sum_n \left[r^{\mu_n} \int_{r_1}^r \frac{dr W_n'(r)}{r^{\mu_n+1}} \right] \left\{ \left[\sigma_n(0) - \frac{im \omega_n(0)}{\mu_n+1} \right] S_n(t) \right. \\ \left. + \left[\dot{\sigma}_n(0) - \frac{im \dot{\omega}_n(0)}{\mu_n+1} \right] T_n(t) \right\} + \frac{im}{\mu_n+1} \frac{W_n(r)}{r} \\ \cdot \left\{ \omega_n(0) f(t) + \dot{\omega}_n(0) g(t) \right\}; \end{aligned} \quad (C35)$$

$$\begin{aligned} \beta(r, t) = \sum_n \left[im r^{\mu_n} \int_{r_2}^r \frac{dr W_n(r)}{r^{\mu_n+2}} \right] \left\{ \left[\sigma_n(0) - \frac{im \omega_n(0)}{\mu_n+1} \right] S_n(t) + \left[\dot{\sigma}_n(0) - \frac{im \dot{\omega}_n(0)}{\mu_n+1} \right] T_n(t) \right\} \\ - \frac{m^2}{\mu_n+1} \frac{1}{r} \int_{r_3}^r \frac{dr W_n(r)}{r} \left\{ \omega_n(0) f(t) + \dot{\omega}_n(0) g(t) \right\} + \\ \frac{1}{r} \int_{r_3}^r dr r W_n(r) \left\{ \omega_n(0) f(t) + \dot{\omega}_n(0) g(t) \right\} \end{aligned} \quad (C36)$$

If the arbitrary lower limits on the integrals are chosen so that

$$(\mu_n+1) W_n'(r_1) = r_1 W_n(r_1) \left[\mu_n+1 + \frac{m^2}{r_1^2} \right]; \quad (C37)$$

$$r_2 W_n'(r_2) = -W_n(r_2) \left[\mu_n+1 - r_2^2 \right]; \quad (C38)$$

$$W_n'(r_3) = r_3 W_n(r_3), \quad (C39)$$

then

$$\mathbf{E} = \nabla \times \tilde{\mathbf{E}} \quad (C40)$$

and

$$\sigma = \nabla \cdot \underline{\xi}. \quad (C41)$$

Thus the general solution to (29) can be written

$$\underline{\xi} = \underline{\xi} + \nabla \phi, \quad (C42)$$

where

$$\nabla^2 \phi = 0. \quad (C43)$$

The m^{th} component of ϕ satisfies

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi^m}{\partial r} \right) - \frac{m^2}{r^2} \phi^m = 0, \quad (C44)$$

whence

$$\phi^m(r, t) = \phi_+^m \left(\frac{r}{r_+} \right)^m + \phi_-^m \left(\frac{r_-}{r} \right)^m. \quad (C45)$$

The time dependence is obtained by substituting (C45) in (C29), whereupon the right hand side vanishes:

$$f^2 \phi_{\pm}^m - (1 \mp m) \phi_{\pm}^m = 0 \quad (C46)$$

This equation is of the same form as (C25), so we can immediately write down the solutions. For consistency with our previous choices, we set

$$p_{\pm}^m(t) = \bar{\phi} \left[\frac{1}{2}(1 \mp m), \frac{1}{2}; \ln f \right], \quad (C47)$$

$$q_{\pm}^m(t) = \dot{\bar{f}} \bar{\phi} \left[1 \mp \frac{1}{2}m, \frac{3}{2}; \ln f \right]. \quad (C48)$$

These satisfy

$$p_{\pm}(0) = 1, \quad \dot{p}_{\pm}(0) = 0; \quad (C49)$$

$$q_{\pm}(0) = 0, \quad \dot{q}_{\pm}(0) = 1. \quad (C50)$$

For $m=2k$, q_+ reduces to f multiplied by a polynomial in $\ln f$ of order k .

Since p_- and q_- both diverge the same way at large t , it is useful to introduce in their place two new solutions which behave asymptotically like Ψ , the standard confluent hypergeometric function of the second kind:

$$\left. \begin{array}{l} P(t) \\ Q(t) \end{array} \right\} = p_-(t) \mp \frac{\sqrt{2} \Gamma(m+1) q_-(t)}{\Gamma(m+\frac{1}{2})} \quad (C51)$$

For large argument we have

$$P(t) \sim (\ln f)^{-\frac{1}{2}(m+1)}, \quad t \rightarrow \infty; \quad (C52)$$

$$Q(t) \sim (\ln f)^{-\frac{1}{2}(m+1)}, \quad t \rightarrow \infty. \quad (C53)$$

All the other asymptotic results we require follow from the general formula

$$\Phi(a, c; z) \sim \frac{1}{\Gamma(a)} e^z z^{a-c}$$

as $z \rightarrow \infty$.

(C54)

APPENDIX D. A BOUND ON THE FUNCTIONS S_n, T_n

Given $\mu > 0$, let $U(t)$ and $h(t)$ satisfy

$$f^2 \ddot{U} + \mu U = 0; \quad (D1)$$

$$f^2 \ddot{h} - h = 0, \quad (D2)$$

where f is defined in Appendix A, and let

$$U(0) = h(0) = a > 0; \quad (D3)$$

$$\dot{U}(0) = \dot{h}(0) = b > 0. \quad (D4)$$

where $a + b = 1$. Then the following inequalities hold for $0 < t < \infty$:

$$|U(t)| \leq h(t) \leq \sqrt{\frac{\pi}{2}} f(t). \quad (D5)$$

Proof: (i) Assume $a=1, b=0$. Then $h=f$. Denote the solution of (D1) with these initial conditions by U_1 . Define

$$G[U] = \frac{\mu U^2}{f^2} + \dot{U}^2 > 0. \quad (D6)$$

Since

$$\dot{G} = \frac{-2\mu U^2 \dot{f}}{f^3} < 0, \quad (D7)$$

we have

$$\frac{\mu U^2}{f^2} + \dot{U}^2 \leq \mu U^2(0) + \dot{U}(0)^2, \quad (D8)$$

or for $U = U_1$,

$$U_1^2 < f^2. \quad (D9)$$

Hence (D5) holds for case (i).

* Proof supplied by Richard Beals.

(ii) Assume $a=0$, $b=1$. Then $h=g$, defined in Eq. (C2). Denote the appropriate solution of (D1) by U_2 . From (D8),

$$\dot{U}_2^2 < U_2^2(0) = 1, \quad (D10)$$

whence

$$|U_2| \leq t. \quad (D11)$$

But from Eq. (C2),

$$t \leq g < \sqrt{\frac{\pi}{2}} f, \quad (D12)$$

which demonstrates (D5) for case (ii). The general result follows, since for $a > 0$, $b > 0$,

$$\begin{aligned} |U| &= |aU_1 + bU_2| \leq a|U_1| + b|U_2| \\ &\leq a f + b g = h \leq \sqrt{\frac{\pi}{2}} f. \end{aligned} \quad (D13)$$

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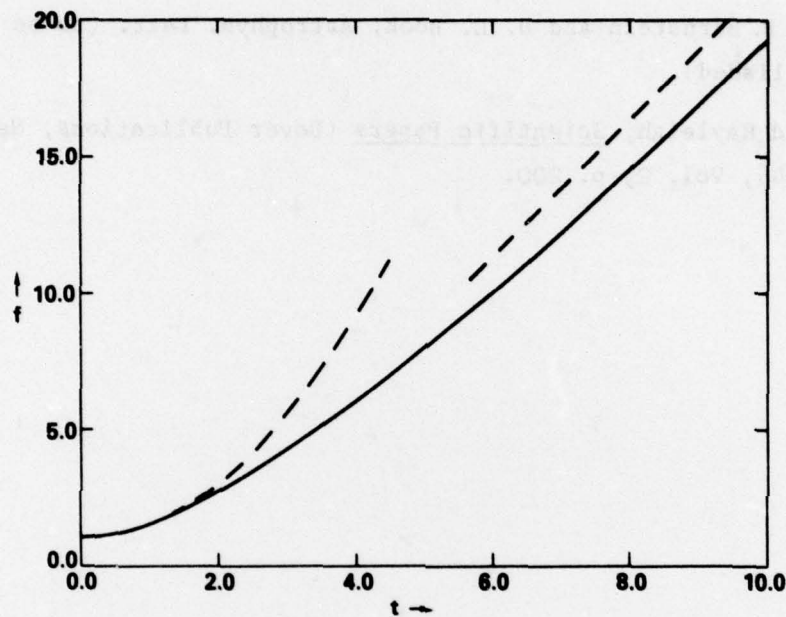


Fig. 1 — Numerical solution of Eq. (10) for $f(t)$ (solid curve), with first order power series and asymptotic approximations [Eqs. (A22) and (A23), upper and lower broken curves, respectively] shown for comparison.

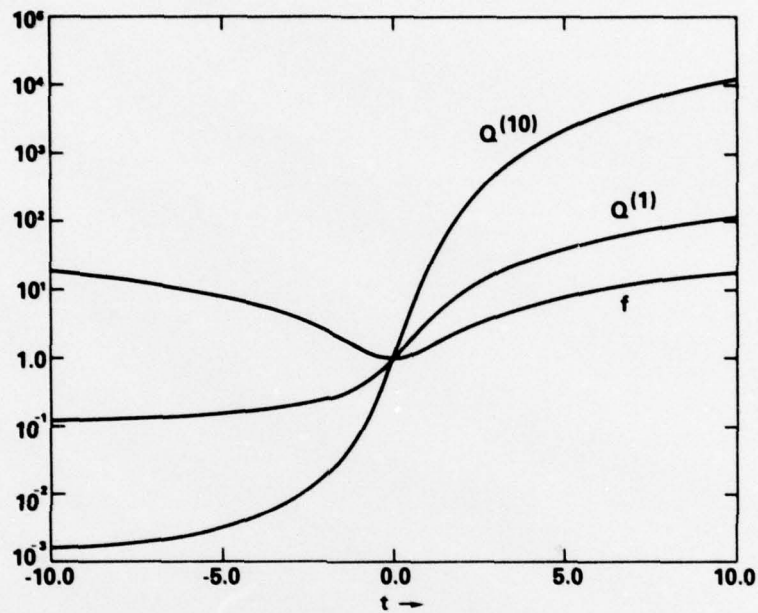


Fig. 2 — $Q^m(t)$ [defined by Eq. (C51)] for $m = 1$ and $m = 10$, obtained by numerical solutions of Eq. (C46), with f plotted for comparison.

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